Heating the $O(N)$ nonlinear sigma model*

Harmen J. Warringa

Department of Physics and Astronomy, Vrije Universiteit,
De Boelelaan 1081, 1081 HV Amsterdam, The Netherlands

February 1, 2008

Abstract

The thermodynamics of the $O(N)$ nonlinear sigma model in $1+1$ dimensions is studied. We calculate the finite temperature effective potential in leading order in the $1/N$ expansion and show that at this order the effective potential can be made finite by temperature independent renormalization. We will show that this is not longer possible at next-to-leading order in $1/N$. In that case one can only renormalize the minimum of the effective potential in a temperature independent way, which gives us finite physical quantities like the pressure.

---

*Seminar talk given at the 43st Cracow School of Theoretical Physics, 30 May - 8 June 2003, Zakopane, Poland

1 Introduction

The nonlinear sigma model is a scalar field theory with an $O(N)$ symmetry. It is described by a Lagrangian density which only consists of a kinetic term,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_i\partial^\mu\phi_i \ , \tag{1}$$

and a constraint which enforces all the ϕ fields to lie on a $N-1$ sphere:

$$\phi_i(x)\phi_i(x) = N/g^2 \qquad i = 1 \ldots N \ . \tag{2}$$

This model has some nice features in $1+1$ dimensions, which makes it interesting to study as a toy model for QCD. First it is renormalizable. Furthermore it is asymptotically free, such that at very high temperatures it approaches a free field theory. The model also has a dynamically generated mass for the ϕ fields. If $N = 3$ the model has instanton solutions. Finally, for $N = 2$ we recover a free field theory, which can be used as a check of the calculations.

In this article we will study the thermodynamical properties of the nonlinear sigma model. In particular we will calculate the pressure. In Sec. 2 we briefly discuss some aspects of thermal field theory. In Sec. 3 we calculate the pressure in the weak-coupling expansion. In Sec. 4, we calculate the effective potential and pressure to leading order in the $1/N$ expansion. The next-to-leading order (NLO) correction is discussed in Sec. 5.

2 The pressure in a field theory

In this section we briefly review how one calculates the pressure in a thermal field theory. For a more complete introduction see Refs. [1, 2].

In classical statistical mechanics one can derive all thermodynamic quantities from the partition function. The partition function $\mathcal{Z}$ is given by

$$\mathcal{Z} = \sum_n \left\langle n \left| \exp[-\beta\hat{H}] \right| n \right\rangle \ , \tag{3}$$

where the sum is over all eigenstates of the Hamiltonian $\hat{H}$ and $\beta = 1/T$, the inverse temperature. For example the pressure $\mathcal{P}$ is given by

$$\mathcal{P} = \frac{1}{\beta}\frac{\partial \log \mathcal{Z}}{\partial V} \ . \tag{4}$$

We next express the partition function in terms of fields. The easiest way to do this is to consider a transition matrix element in ordinary field theory. One can write such a transition element in terms of a path integral in the following way

$$\left\langle \phi_f \left| \exp[-i(t_f - t_i)\hat{H}] \right| \phi_i \right\rangle = \int \mathcal{D}\phi \ \exp\left[i\int_{t_i}^{t_f} \mathrm{d}t \int \mathrm{d}^d x \, \mathcal{L}(\phi) \right] \ , \tag{5}$$

where $\mathcal{L}$ is a Lagrangian density which has a Minkowskian metric and does not have derivative interactions. Now if one makes the identification $t = -i\tau$ one finds

$$\left\langle \phi_f \left| \exp[-\beta\hat{H}] \right| \phi_i \right\rangle = \int \mathcal{D}\phi \ \exp\left[-\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^d x \, \mathcal{L}(\phi) \right] \ , \tag{6}$$

where we from now on denote the zero component of a $(d+1)$-vector by τ and hence use a Euclidean metric. The last equation enables us to write the partition function in terms of a path integral,

$$\mathcal{Z} = \int \mathcal{D}\phi \, \exp\left[-\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^d x \, \mathcal{L}(\phi) \right]_{\phi(\tau=0)=\phi(\tau=\beta)} , \tag{7}$$

where one implicitly integrates over all states which obey the periodicity condition $\phi(\tau=0,\vec{x}) = \phi(\tau=\beta,\vec{x})$. So equilibrium thermal field theory is in essence a Euclidean field theory where one dimension (τ) is compactified to a circle. As a consequence, the Fourier transform of a field becomes a sum over modes,

$$\phi(\tau,\vec{x}) = \frac{1}{\beta}\sum_n \int \frac{\mathrm{d}^d k}{(2\pi)^d} e^{i\omega_n\tau + i\vec{k}\cdot\vec{x}} \tilde{\phi}(k) \equiv \sum\!\!\!\!\!\!\!\int_K e^{i\omega_n\tau + i\vec{k}\cdot\vec{x}} \tilde{\phi}(k) , \tag{8}$$

where $\omega_n = 2\pi n T$. This implies that in a loop diagram one should not take the integral over internal momentum but rather the sum-integral $\sum\!\!\!\!\!\!\int$.

Now for example the partition function of the nonlinear sigma model is given by

$$\mathcal{Z} = \int \prod_{i=1}^N \mathcal{D}\phi_i \, \prod_x \delta(\phi_i(x)\phi_i(x) - N/g^2) \exp\left[-\int_0^\beta \mathrm{d}\tau \int \mathrm{d}x \, \mathcal{L}(\phi) \right] , \tag{9}$$

where from now on we work in one spatial dimension. To obtain the pressure we have to calculate $\mathcal{Z}$. We will follow two paths. The first one is making an expansion around $g^2 = 0$. This will only give us the leading term of the pressure. The second way is an expansion in $1/N$ which will generates additional contributions which are non-analytical in g^2.

3 The pressure in the weak-coupling expansion

One can get rid of the constraint by integrating out one of the ϕ fields, which results in

$$\mathcal{Z} = \int \prod_{i=1}^{N-1} \mathcal{D}\pi_i \prod_x \theta\left(N/g^2 - \pi_i\pi_i\right) \exp\left[-\int_0^\beta \mathrm{d}\tau \int \mathrm{d}x \, \mathcal{L}_{\mathrm{eff}}(\pi) \right] , \tag{10}$$

where $\theta(x)$ is the step function and the effective Lagrangian density $\mathcal{L}_{\mathrm{eff}}$ is given by

$$\mathcal{L}_{\mathrm{eff}}(\pi) = \frac{1}{2}\partial_\mu\pi_i\partial^\mu\pi_i + \frac{g^2}{2}\frac{(\pi_i\partial_\mu\pi_i)^2}{N - g^2\pi_i\pi_i} - \frac{1}{2}\beta V \log\left(N/g^2 - \pi_i\pi_i\right) . \tag{11}$$

For small values of g^2 the $\theta(x)$ function is only vanishing when $\pi(x)$ is large. Since large values of π give a small contribution to the path integral we approximate $\theta(N/g^2 - \pi_i\pi_i) \approx 1$ which gives

$$\mathcal{Z} = \int \prod_{i=1}^{N-1} \mathcal{D}\pi_i \exp\left[-\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^d x \, \mathcal{L}_{\mathrm{eff}}(\pi) \right] . \tag{12}$$

We will not calculate $\mathcal{Z}$ but rather $\frac{1}{\beta V}\log\mathcal{Z}$, where V is the volume of our 1 dimensional space. Because $\log\mathcal{Z}$ is an extensive quantity, *i.e.* it is linear in V, the pressure is equal to $\frac{1}{\beta V}\log\mathcal{Z}$. Since in general $\frac{1}{\beta V}\log\mathcal{Z}$ does not vanish at zero temperature, we subtract the zero temperature contribution to normalize the pressure to zero at zero temperature.

If $g^2 = 0$ it can be seen from $\mathcal{L}_{\rm eff}$ that one has $N-1$ noninteracting π fields. Hence it is easy to show that leading term is equal to the pressure of $N-1$ free fields

$$\mathcal{P} = -\frac{N-1}{2}\left[\sum\!\!\!\!\!\!\int_K \log(K^2) - \int_K \log(K^2)\right] = (N-1)\frac{\pi}{6}T^2 \, , \tag{13}$$

where $K = (\omega_n, k\,)$ is a Euclidean two vector and we defined

$$\int_K \equiv \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \, . \tag{14}$$

By calculating the loop diagrams, one can show that up to and including order g^4 one only finds the pressure of a free gas in $d = 1+1$ [3, 4]. However one finds corrections to the free pressure in a $1/N$ expansion. This may indicate that the pressure is completely non-analytical in g^2.

4 The effective potential in leading order in $1/N$

Another way to implement the constraint on the ϕ fields is by using a Lagrange multiplier field which we will denote by α. This gives the following expression for the partition function,

$$\begin{aligned}\mathcal{Z} = \int \prod_{i=1}^{N} \mathcal{D}\phi_i \mathcal{D}\alpha \exp\Bigg\{ &-\frac{1}{2}\int_0^\beta \mathrm{d}\tau \int \mathrm{d}x\, \partial_\mu\phi_i\partial^\mu\phi_i \\ &-\frac{1}{2}\int_0^\beta \mathrm{d}\tau \int \mathrm{d}x\, \alpha(x)[\phi_i(x)\phi_i(x) - N/g^2]\Bigg\} \, .\end{aligned} \tag{15}$$

In this way the action still is quadratic in the ϕ fields, so one can easily integrate them out. This gives

$$\mathcal{Z} = \int \mathcal{D}\alpha \exp\left\{ -S[\alpha] + \frac{N}{2g^2}\int_0^\beta \mathrm{d}\tau \int \mathrm{d}x\, \alpha(x)\right\} \, , \tag{16}$$

where

$$S[\alpha] = \frac{N}{2}\mathrm{Tr}\log[-\partial^2 + \alpha(x)] \, . \tag{17}$$

The pressure is equal to the minimum of the effective potential, which one can calculate by expanding the α field around its vacuum expectation value m^2. By considering the propagator of the ϕ fields, one can show that to leading order in $1/N$, m is equal to the physical mass of the ϕ fields. This is, however, not longer the case at NLO, [5, 6]. The effective potential can be obtained from the effective action by division by βV. To calculate the effective potential we write $\alpha = m^2 + \tilde{\alpha}/\sqrt{N}$ and expand the action around m^2 [7],

$$S[\alpha] = \frac{N}{2}\mathrm{Tr}\log[-\partial^2 + m^2] + \frac{\sqrt{N}}{2}\mathrm{Tr}\left(\frac{1}{-\partial^2 + m^2}\tilde{\alpha}\right) + \frac{1}{4}\mathrm{Tr}\left(\frac{1}{-\partial^2 + m^2}\tilde{\alpha}\right)^2 + \mathcal{O}(1/\sqrt{N}) \, . \tag{18}$$

From this equation it can easily be seen that the effective potential can be calculated in a $1/N$ expansion. The leading order effective potential is given by the classical action. The corrections are obtained by integrating over the $\tilde{\alpha}$ field.

To calculate the leading order effective potential we introduce a momentum cutoff Λ and subtract m and T-independent constants from the effective potential. This subtraction will not

change the physics, since it only shifts the whole effective potential by a constant. One finds for the effective potential at leading order in $1/N$

$$\mathcal{V}(m^2) = \frac{Nm^2}{2g_b^2} - \frac{N}{2}\left[\sum\!\!\!\!\!\!\int_P \log(P^2+m^2) - \int_P \log(P^2)\right] \tag{19}$$

$$= \frac{N}{2}\left[\frac{m^2}{g_b^2} - \frac{m^2}{4\pi}\left(1+\log\frac{\Lambda^2}{m^2}\right) + \frac{T^2}{4\pi}J_0(\beta m)\right] , \tag{20}$$

where g_b is the bare coupling constant. $J_0(\beta m)$ is given by

$$J_0(\beta m) = \frac{8}{T^2}\int_0^\infty \mathrm{d}p\, \frac{p^2 n(\omega_p)}{\omega_p} , \tag{21}$$

where $n(x) = 1/(e^x - 1)$ and $\omega_p^2 = p^2 + m^2$. One is able to renormalize the leading order effective potential in a temperature independent way by replacing $g_b^2 \to Z_{g^2} g^2(\mu)$ where

$$\frac{1}{Z_{g^2}} = 1 + \frac{g^2}{4\pi}\log\frac{\Lambda^2}{\mu^2} . \tag{22}$$

and $g^2 = g^2(\mu)$. From this equation it follows that the β-function of g^2 is given by

$$\beta(g^2) \equiv \mu\frac{\mathrm{d}g^2(\mu)}{\mathrm{d}\mu} = -\frac{g^4(\mu)}{2\pi} . \tag{23}$$

The leading order β-function is exact in g^2. Since the β-function is negative, g^2 approaches zero for large values of μ. This shows that the theory is asymptotically free.

With use of the renormalization of the coupling constant one finds the following finite expression for the effective potential

$$\mathcal{V}(m^2) = \frac{N}{2}\left[\frac{m^2}{g^2} - \frac{m^2}{4\pi}\left(1+\log\frac{\mu^2}{m^2}\right) + \frac{1}{4\pi}T^2 J_0(\beta m)\right] . \tag{24}$$

One can easily show that the effective potential is independent of the renormalization scale μ. This is expected since the choice of μ is completely arbitrary.

To obtain the pressure, one has to minimize the effective potential with respect to m^2. Minimization gives the so-called gap equation

$$\frac{1}{g^2} = \sum\!\!\!\!\!\!\int_P \frac{1}{P^2+m^2} = \frac{1}{4\pi}\log\left(\frac{\mu^2}{m^2}\right) + \frac{1}{4\pi}J_1(\beta m) \equiv \frac{1}{4\pi}\log\left(\frac{\mu^2}{\bar{m}^2}\right) , \tag{25}$$

where $J_1(\beta m)$ is defined by

$$J_1(\beta m) = 4\int_0^\infty \mathrm{d}p\, \frac{n(\omega_p)}{\omega_p} . \tag{26}$$

The solution of the gap equation determines the leading order physical mass of the ϕ fields as a function of temperature. At $T = 0$ one can solve this equation to show that the mass is completely non-analytical in g^2,

$$m_{T=0} = \mu\exp\left(-\frac{2\pi}{g^2}\right) . \tag{27}$$

We can use Eq. (27) to normalize the minimum of the effective potential at $T = 0$ to be zero which gives

$$\mathcal{V}(m^2) = \frac{N}{2}\left[\frac{m^2}{g^2} - \frac{m^2}{4\pi}\left(1+\log\frac{\mu^2}{m^2}\right) + \frac{1}{4\pi}T^2 J_0(\beta m) + \frac{m_{T=0}^2}{4\pi}\right] . \tag{28}$$

The effective potential as a function of m for different temperatures is shown in Fig. (1), for the arbitrary choice $g^2(\mu = 500) = 10$. The quantities T, m, μ, $\mathcal{V}/T$ and $\mathcal{P}/T$ are all in the same arbitrary units. The solid curve which is the minimum of the effective potential is equal to the pressure.

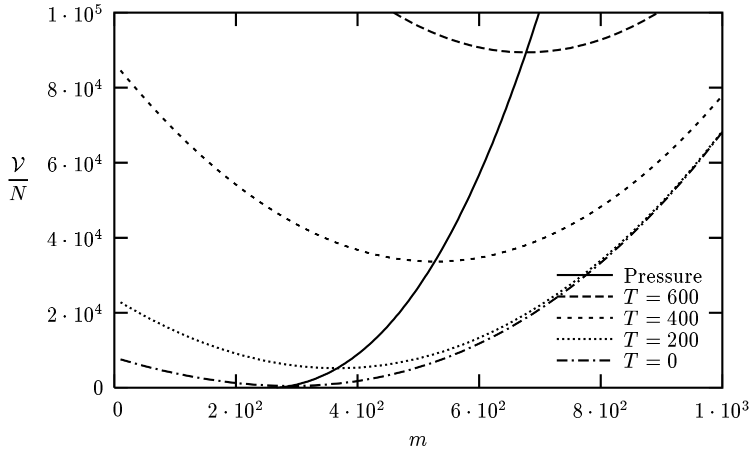

Figure 1: The leading order effective potential as function of m for different temperatures with $g^2(\mu = 500) = 10$.

5 Next-to-leading order correction in $1/N$

The term linear in α in Eq. (18) gives no contribution to the effective potential since it gives rise to a tadpole [8]. The first $1/N$ correction to the effective potential stems from the last term of Eq. (18). By going to momentum space one can show that the correction is given by

$$\mathcal{V}_1(m^2) = -\frac{1}{2}\sum\!\!\!\!\!\!\!\int_P \log\left[\sum\!\!\!\!\!\!\!\int_Q \frac{1}{Q^2+m^2}\frac{1}{(P+Q)^2+m^2}\right] . \tag{29}$$

We calculated this correction in Ref. [6]. In the limit $\Lambda \to \infty$, one obtains

$$\mathcal{V}_1(m^2) \;=\; -\frac{1}{8\pi}\left(\Lambda^2 \ln\ln\frac{\Lambda^2}{\bar{m}^2} - \bar{m}^2\,\mathrm{li}\,\frac{\Lambda^2}{\bar{m}^2}\right) - \frac{m^2}{4\pi}\left(\ln\ln\frac{\Lambda^2}{\bar{m}^2} - \ln\frac{\Lambda^2}{4m^2}\right) + F(m,T) \quad , \tag{30}$$

where $\bar{m}$ is defined in Eq. (25). In (30), we have subtracted m and T-independent constants and dropped terms that vanish in the limit $\Lambda \to \infty$. $F(m,T)$ is a finite term and the logarithmic integral li is defined by

$$\mathrm{li}(x) = \mathcal{P}\int_0^x \mathrm{d}t\,\frac{1}{\log t} , \tag{31}$$

where $\mathcal{P}$ stands for principal value. The first two terms of Eq. (30) are problematic. It is impossible to remove these divergences by renormalizing g^2 in a temperature independent way or by subtracting m and T-independent constants. However this is possible at the minimum of the effective potential. At the minimum, one can use the leading order gap equation, Eq. (25), to show that $\bar{m}$ is independent of T. So one could add

$$\frac{\Lambda^2}{8\pi}\left\{\ln\frac{4\pi}{g_b^2} - \exp\left(-\frac{4\pi}{g_b^2}\right)\mathrm{li}\left[\exp\left(\frac{4\pi}{g_b^2}\right)\right]\right\} \tag{32}$$

to the effective potential which yields an effective potential that can be renormalized at the minimum. Using this renormalization at the minimum we have calculated the pressure $\mathcal{P}$ as a function of N. The result is depicted in Fig. (2). One clearly sees a crossover which is not a phase transition. This is in accordance with the Mermin-Wagner-Coleman theorem [9, 10] which forbids spontaneous breakdown of a continuous symmetry in $1+1$ dimensions. The figure furthermore shows that the $1/N$ expansion is relatively good since the corrections are really of order $1/N$. Finally it can be seen from the figure that the theory is asymptotically free, because in the limit $T \to \infty$ the pressure approaches the pressure of a free gas, Eq. (13).

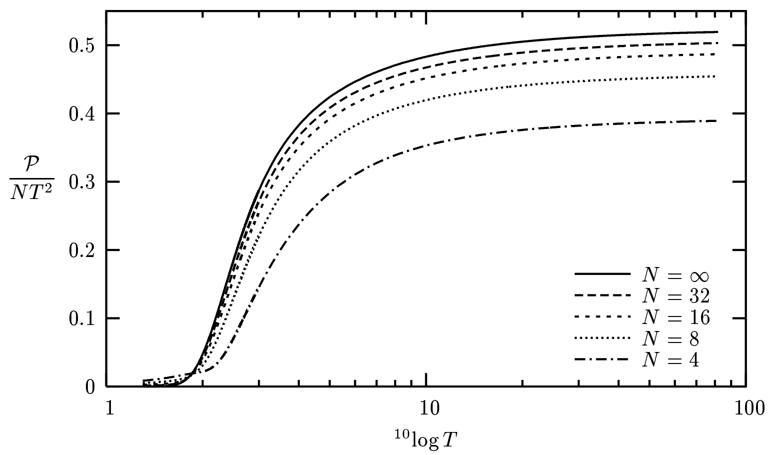

Figure 2: Pressure $\mathcal{P}$ normalized to NT^2 as a function of temperature for different values of N with $g^2(\mu = 500) = 10$ [6].

6 Summary and Conclusions

We find that the pressure of the nonlinear sigma model in the weak-coupling expansion through order g^4 only consist of the free term. Furthermore, we showed that in a $1/N$ expansion we can renormalize the leading order effective potential in a temperature independent way. This is, however, impossible for the effective potential at next-to-leading order in $1/N$. In that case one can only renormalize in a temperature-independent way a physical quantity, like the pressure.

Acknowledgments

This work has been carried out in collaboration with Jens O. Andersen and Daniël Boer.

References

[1] J. I. Kapusta, *Finite-temperature field theory*, Cambridge University Press (1989)

[2] M. Le Bellac, *Thermal field theory*, Cambridge University Press, (2000)

[3] M. Dine and W. Fischler, Phys. Lett. B **105**, 207 (1981).

[4] J. O. Andersen, D. Boer and H. J. Warringa, in preparation.

[5] H. Flyvbjerg, Phys. Lett. B **245**, 533 (1990).

[6] J. O. Andersen, D. Boer and H. J. Warringa, arXiv:hep-ph/0309091.

[7] V. A. Novikov, M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Phys. Rept. **116**, 103 (1984).

[8] J. Zinn-Justin, *Quantum Field Theory And Critical Phenomena*, Oxford University press (1996), 3rd edition.

[9] N. D. Mermin and H. Wagner, Phys. Rev. Lett. **17**, 1133 (1966).

[10] S. R. Coleman, Commun. Math. Phys. **31**, 259 (1973).